On the Complexity of Planar Regular Covering

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joint work with Jiří Fiala,
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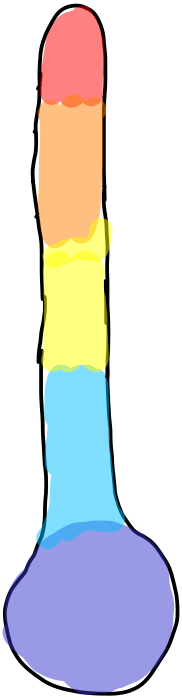
In mathematics, many papers study degree of similarity of objects.



isomorphisms

homomorphisms

most objects are not similar

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isomorphisms

locally the same coverings

homomorphisms

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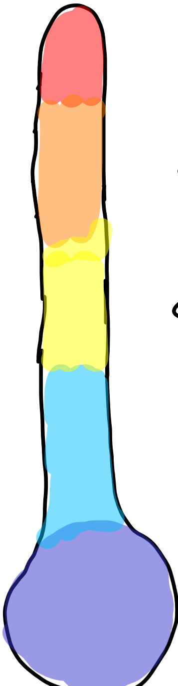


isomorphisms

regular coverings

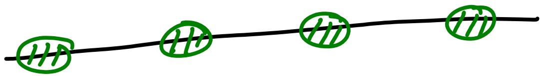
locally the same coverings

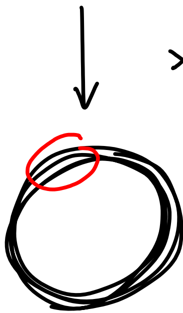
homomorphisms

most objects are not similar

Notion of covering originates in topology.

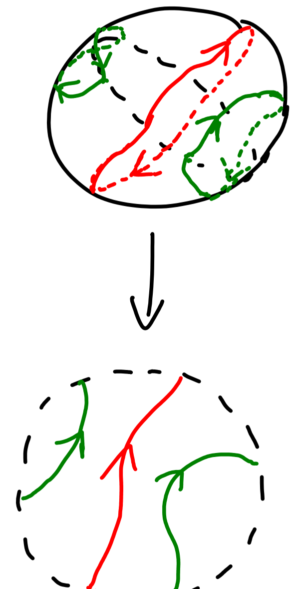
"Two surfaces are locally the same."



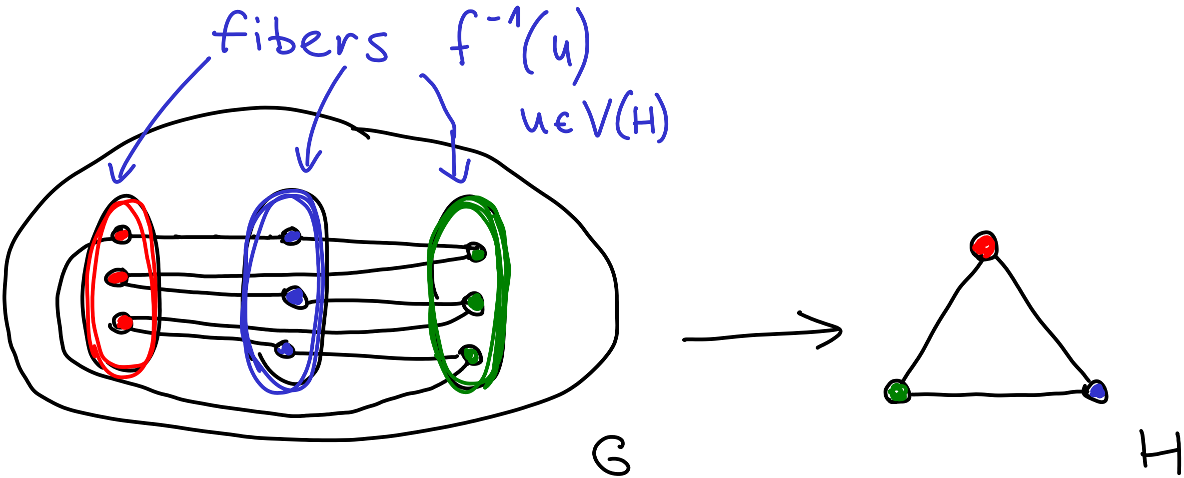


$x \longrightarrow (\cos x, \sin x)$

- real line covers a circle
- sphere covers the proj. plane

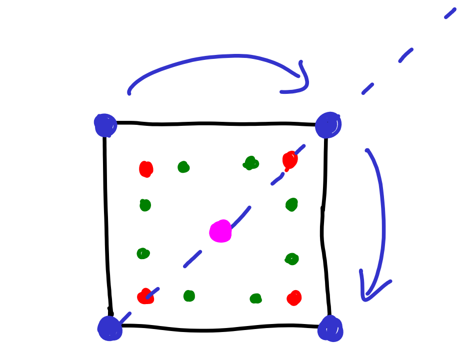


For graphs, G covers H if there exists a locally bijective homomorphism $f: V(G) \to V(H)$ s.t. $f|_{N[u]}$ maps bijectivelly to $N[f(u)]$.



Group actions

- A group G transforms a set S.



S is square

G is D_4

- Orbit of $x \in S$ consists of all images of x under action of G.
- G acts transitively is it has exactly one orbit

Group actions

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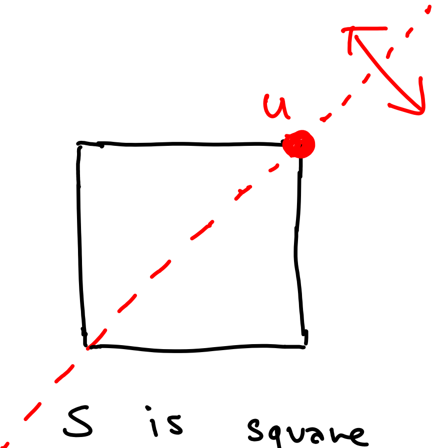


S is square

G is D_4

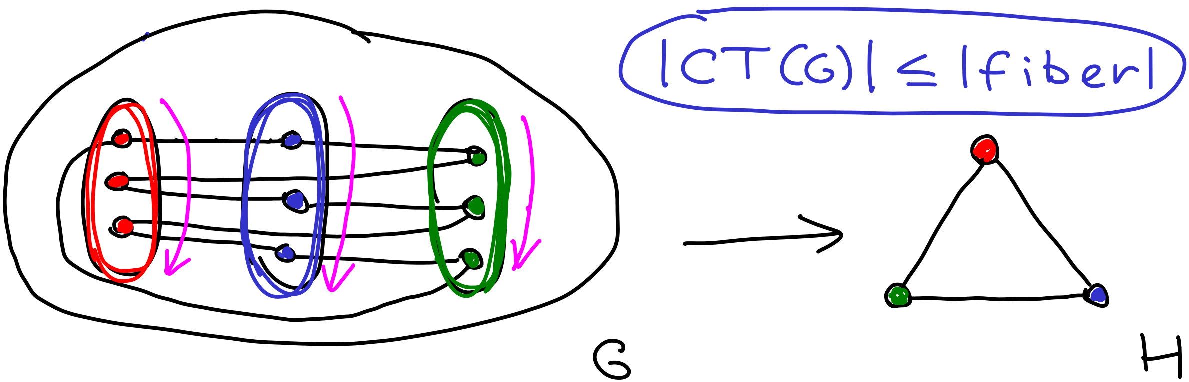
- stabilizer $S(u)$ is a set of actions s.t. $u = u \circ g$

- group acts semiregul. if for $\forall u \in S$ it has only the trivial stabilizer.

Covering transformation group

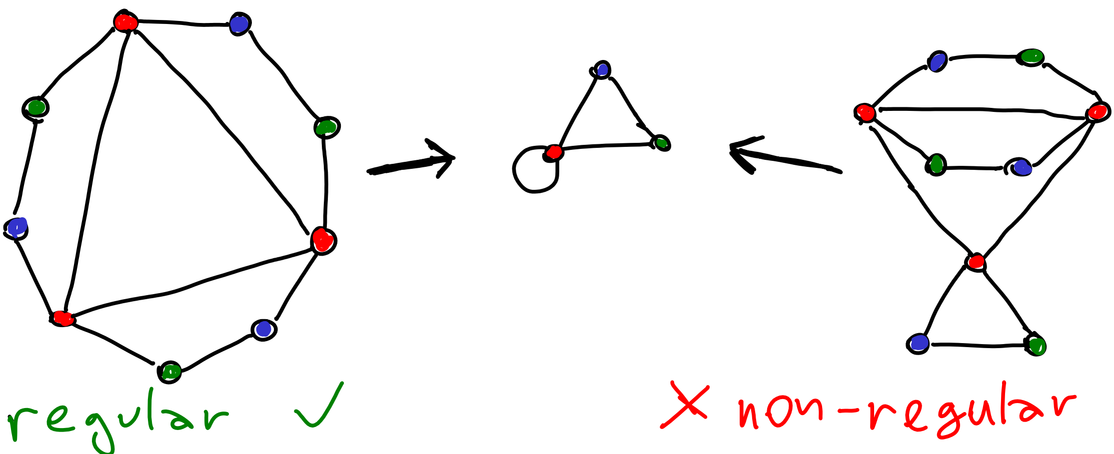
- covering transformations are **fiber-preserving automorphisms**
- they form a **semiregular subgroup** $CT(G) \leq Aut(G)$

$$|CT(G)| \leq |fiber|$$



So what is a regular covering?

Covering is regular if CT acts transitively on every fiber, in other words $|CT(G)| = |\text{fiber}|$.

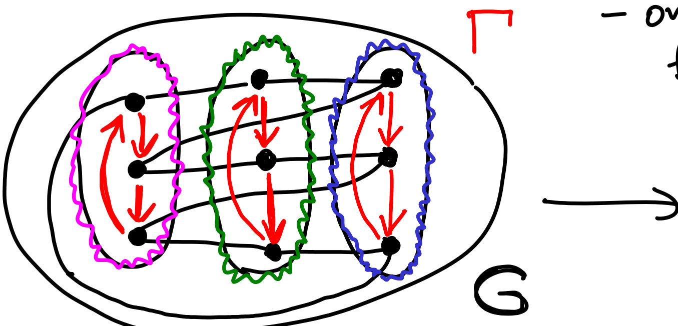


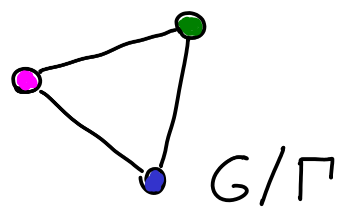
Definition using factorization:

Let Γ be a semiregular subgroup of $\mathrm{Aut}(G)$. Then the factorization

G/Γ defines a regular covering projection.

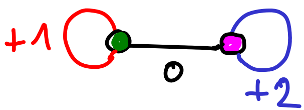
- orbits of Γ are fibers of the covering





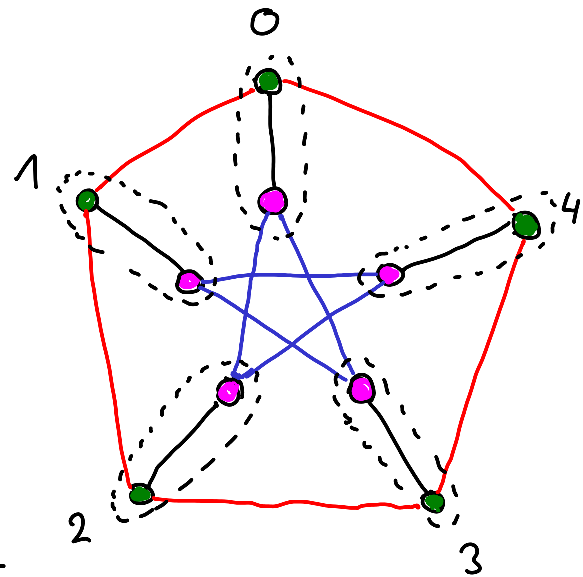
Definition using voltages:

Based on unique walk lifting property.



lift

$CT(G) = \mathbb{Z}_5$

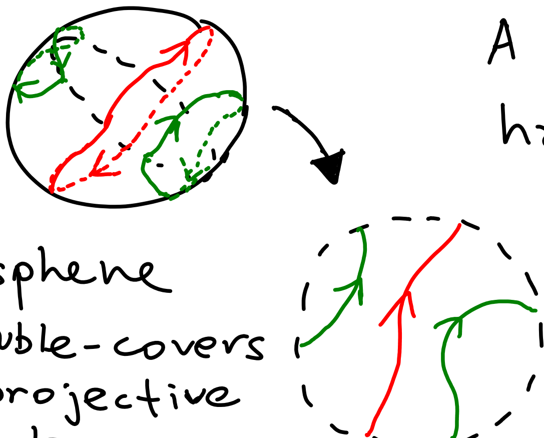


- make $|CT(G)|$ copies of the spanning tree of H
- put remaining edges according to some element of $CT(G)$

What about planarity?

Let the big graph G be planar.

Closely related to Negami's Conj:



sphere double-covers projective plane

A connected graph H has a finite planar cover $\Longleftrightarrow$ H can be embedded into the projective plane.

General case is still open:

- property „H has a planar cover" is closed under minor relation and $Y\Delta$-transformations
- only one forbidden minor $K_{1,2,2,2}$ open.

Negami proved restricted versions:

- for 2-fold covers
- for regular covers

Complexity problem H-COVER

Input: A graph G.

Question: Does G covers H.

There are some results:

Kratochvíl, Proskurowski, Telle, Fiala

- k-reg H is NP-c for $k \geq 3$
- dichotomy for small graphs

The H-PlanarCover problem:

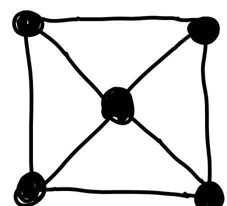
Input: A planar graph G.

Question: Does G covers H.

The problem is trivial for $H = K_2, \ldots$
Non-trivial cases seems equally hard.

Bílka, Jirásek, K., Tancer, Volec. WG'11

Small Open case:



W_4

Idea from last ATCAGC:

The PLANAR REGULAR COVER problem

Input: Graphs G and H,
G planar, H projective.

Question: Does G regularly covers H?

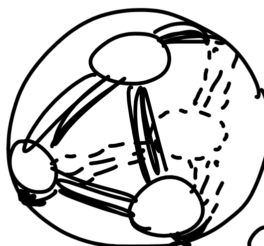
We believe that this problem is solvable in polynomial time.

Let $\mathcal{M}$ be an embedding of G, then $\operatorname{Aut}(G) \geq \operatorname{Aut}(\mathcal{M})$.

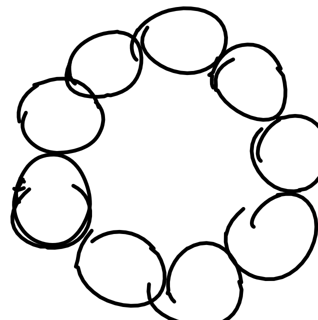
- $\operatorname{Aut}(\mathcal{M})$ is very restricted:
  - groups of platonic solids

    $S_4, \mathbb{Z}_2 \times A_5, \ldots$

  - $\mathbb{Z}_n, D_n$



S_4



$\mathbb{Z}_n$

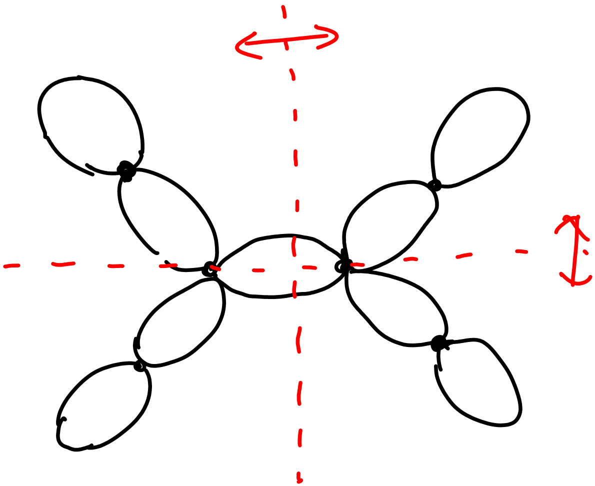
Idea is to compute $\mathrm{Aut}(G)$ and to guess $CT(G)$.

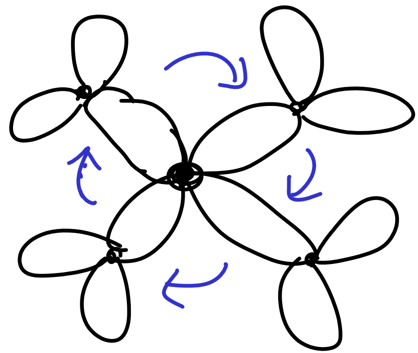
If G is 3-connected, the embedding is unique and $\mathrm{Aut}(G) = \mathrm{Aut}(\mathcal{M})$.

So we can guess $CT(G)$ and to compute $G/CT(G)$ and to test isomorphism with H.

1-connected pieces are easy:

Consider the block tree of G.





Every element of Aut(G) fixes control block or articulation.

Thank You!