MSOL Restricted Contractibility to Planar Graphs

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joint work with J. Abello, J. Kratochvíl and T. Vyskočil

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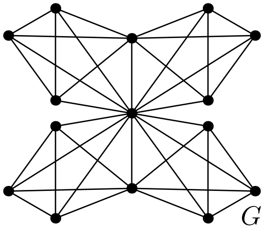

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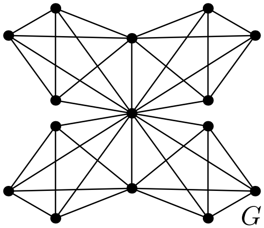
How to make a graph planar?

- By removing some vertices.
- By removing some edges.
- By contracting some edges.



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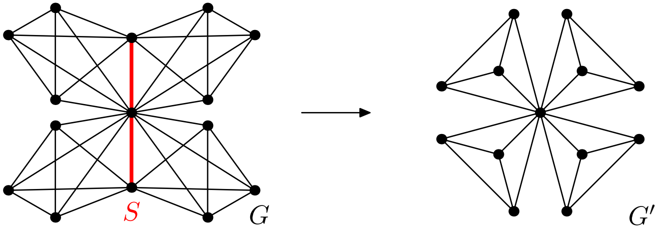
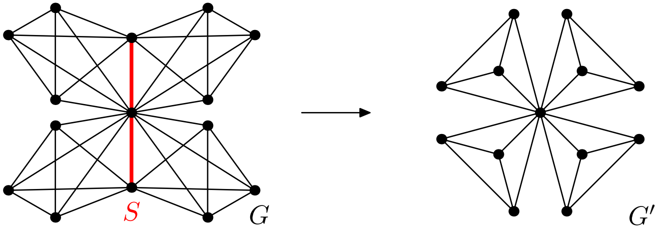


- We want to contract $S \subseteq E(G)$ such that $G' = G \circ S$ is planar.
- Such S is called a planarizing set of G.
- In addition, $|S|$ should be as small as possible.
- A graph G is called k-contractible, if there exists a planarizing set S such that $|S| \leq k$.



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- The following decision problem was considered:

Problem: Contract
Input: A graph G and an integer k.
Output: Does there exist a planarizing set S such that $|S| \leq k$?

- This problem is NP-complete (Asano, Hirata, 1983).

- Here, we show that Contract is solvable in time $\mathcal{O}(n^2 \cdot f(k))$.
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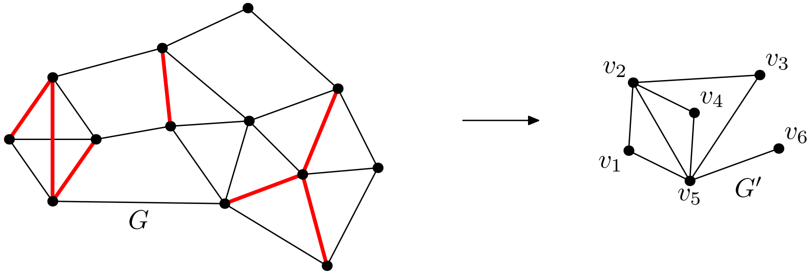
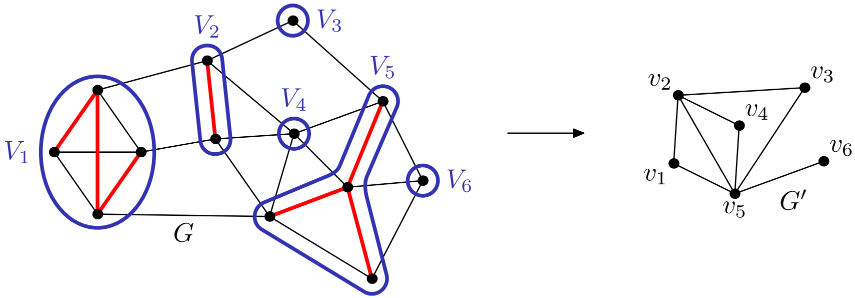


- Edges of minimal S form a forest in G.
- Each connected component is contracted into a single vertex.
- Components of S give connected clusters $V_1, \ldots, V_c$ in G.
- The graph G' is a cluster graph of G.
- CONTRACT: Does there exist at least $n - k$ connected clusters in G, such that the cluster graph G' is planar?



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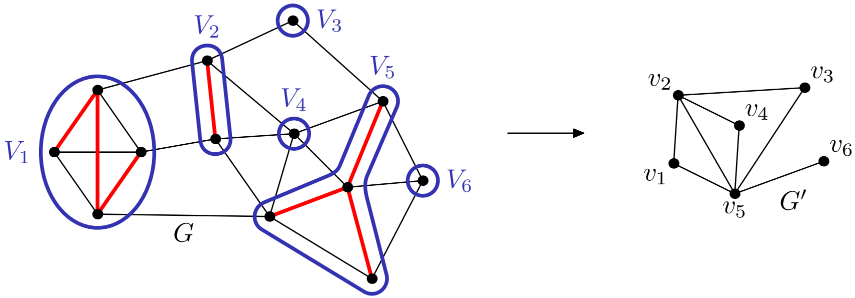

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- It might happen that clusters have very different sizes.
- If a cluster is very large, a lot of information of G is lost in G'.

- We can restrict clusters to be small:

Problem: ℓ-SUBCONTRACT
Input: A graph G, integers k and ℓ.
Output: Does there exist a planarizing set S such that $|S| \leq k$ and each cluster is of size at most ℓ?

- This problem is interesting even for $k = n$.
- For $\ell = 2$, ℓ-SUBCONTRACT forces S to be a matching in G.

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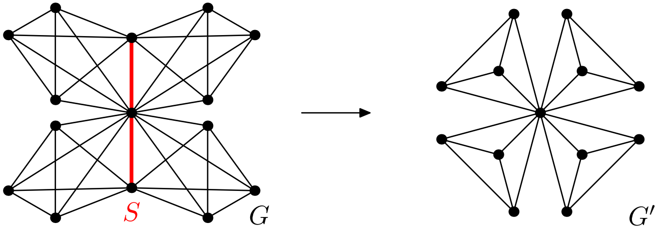
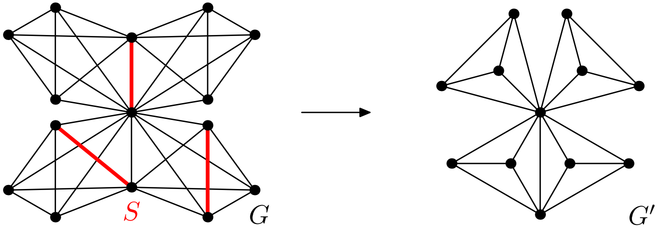
Optimal solution for CONTRACT:



Optimal solution for ℓ-SUBCONTRACT for $\ell = 2$:



- More generally, we restrict S by some MSOL formula $P(S; G)$:

Problem:	P-RESTRICTEDCONTRACT
Input:	A graph G and an integer k.
Output:	Does there exist a planarizing set S such that $\|S\| \leq k$ and $P(S, G)$ is satisfied?

- Some examples:
  - We can restrict sizes of the clusters (ℓ-SUBCONTRACT).
  - We can restrict diameter of the clusters.
  - We can have a set F of forbidden edges.
  - We can restrict the maximum number of clusters.
  - We can ask silly things: "Is G a 3-colorable graph?"

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- A formula $P(S, G)$ is inclusion-closed if

$$P(S,G) = 1 \implies P(S',G) = 1, \quad \forall S' \subseteq S.$$

- A set of edges B of G is called boring if it is not incident with any edge for any inclusion-minimal planarizing set S.

- A formula $P(S, G)$ is boring contraction-closed if

$$P(S,G) = 1 \iff P(S, G \circ B) = 1, \quad \forall \text{ boring } B.$$

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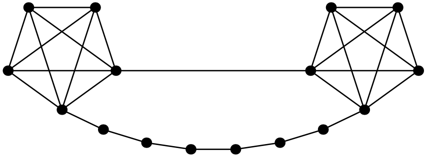

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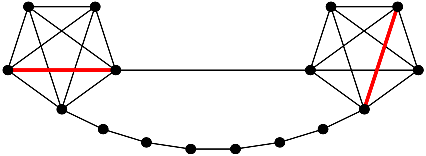

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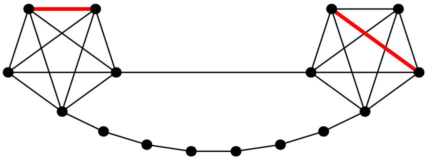

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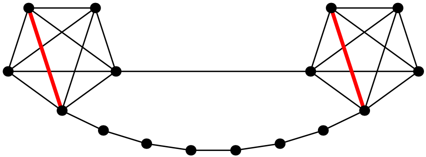

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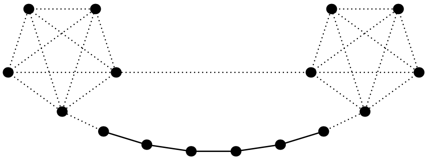

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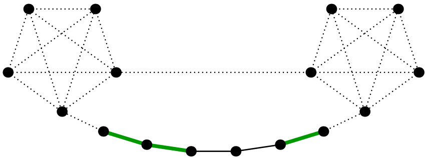

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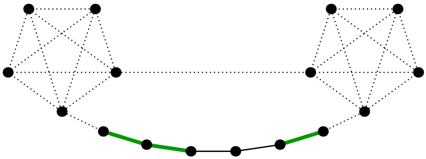

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Theorem

For every inclusion-closed and boring contraction-closed formula $P(S, G)$, P-RESTRICTEDCONTRACT can be solved in time $\mathcal{O}(n^2 \cdot f(k))$.

Remark: Without these assumptions on P, the problem can be hard.

$$P(S, G) \quad = \quad \text{"all edges are forbidden to contract and } G \text{ is a 3-colorable graph".}$$

Then P-RESTRICTEDCONTRACT is NP-hard even for $k = 0$.

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The problem ℓ-SUBCONTRACT is NP-complete for every $\ell \geq 2$.

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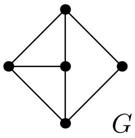
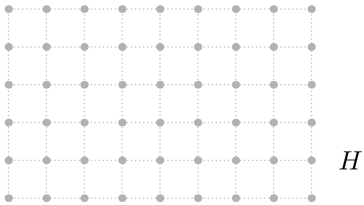
Topological embedding $h : G \hookrightarrow H$:

Consists of $h_V : V(G) \to V(H)$ and $h_E : E(G) \to P(H)$ such that

- Mapping h_V is injective.
- For $e \neq f$ the paths $P(e)$ and $P(f)$ are internally disjoint and share at most one endpoint.
- For $P(uv)$ are $h_V(u)$ and $h_V(v)$ its endpoints. Also $P(uv)$ avoids $h_V(w)$ for every w distinct from u and v.

In other words, a subdivision of G is a subgraph of H.





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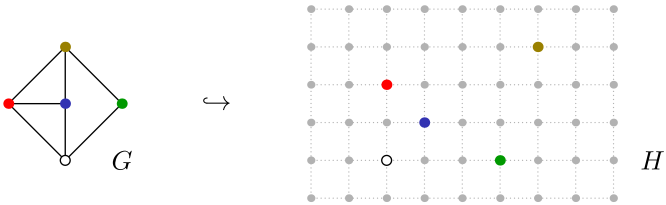

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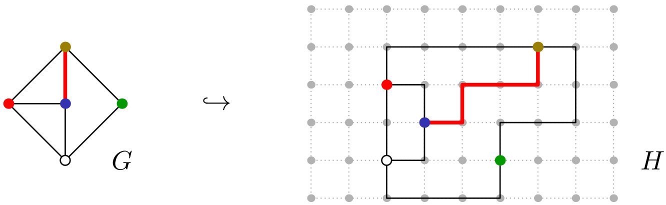

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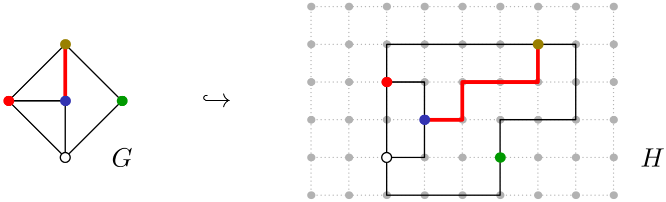
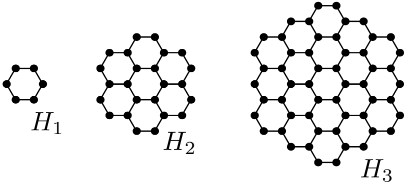


Hexagonal grids:



Theorem (Robertson, Seymour, Boadlaender, Perkovič, Reed)

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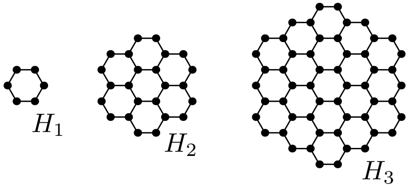

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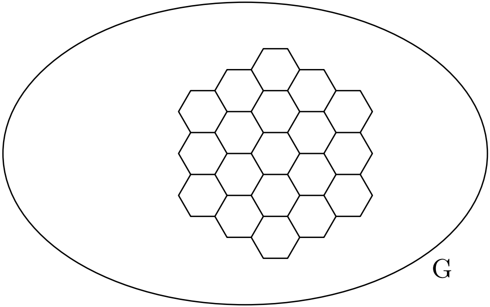
- We have an embedding $h : H_k \hookrightarrow G$.
- Consider interior of the grid and attached components of $G \setminus h(H_k)$, called interior components of $h(H_k)$.
- We denote $h(H_s)$ together with interior components as $h_+(H_k)$.
- The embedding h is flat if $h_+(H_k)$ is a planar graph.



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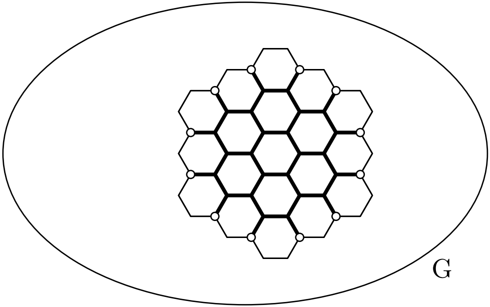

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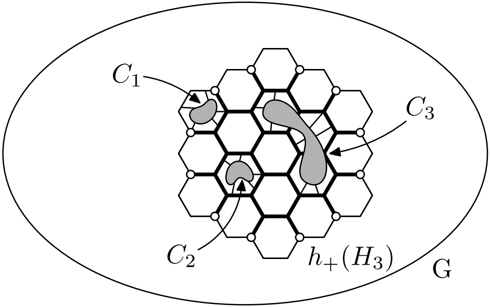

Lemma (Flat Embedding)

Let G be a k-contractible graph. For every r there exists s such that for every $h : H_s \hookrightarrow G$ there exists a subgrid $H_r \subset H_s$ such that embedding $h \restriction H_r$ is flat.

Interior components is small if it is attached to only one cell of the grid, and it is large otherwise.

Plan:

1. Find $H_t \subset H_s$ avoiding large components.
2. Find $H_r \subset H_t$ avoiding non-planar small components.

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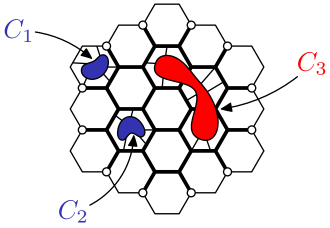


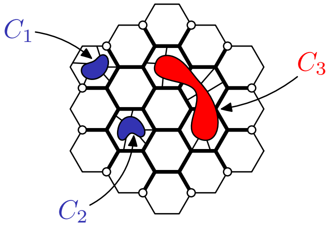
Lemma (Flat Embedding)

Let G be a k-contractible graph. For every r there exists s such that for every $h : H_s \hookrightarrow G$ there exists a subgrid $H_r \subset H_s$ such that embedding $h \restriction H_r$ is flat.

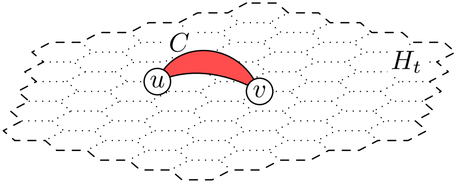
Interior components is small if it is attached to only one cell of the grid, and it is large otherwise.

Plan:

1. Find $H_t \subset H_s$ avoiding large components.
2. Find $H_r \subset H_t$ avoiding non-planar small components.



- Suppose that $h_+(H_t)$ contains a large component.
- Add two layers to the grid.
- Using large component as a bridge, we can embed $K_{3,3}$.
- This $K_{3,3}$ has to be contracted by some edge incident to the grid.
- So for $2k+1$ disjoint embeddings of H_t', at least one does not contain large component.



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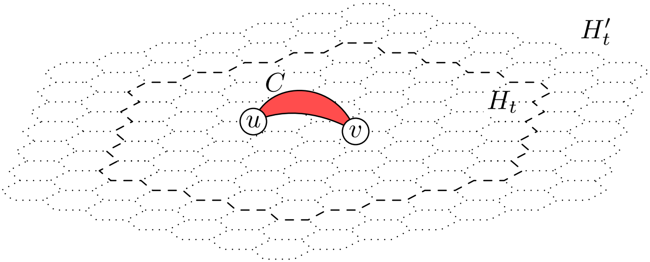

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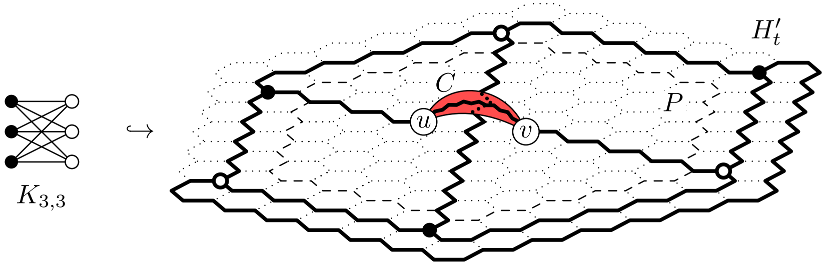

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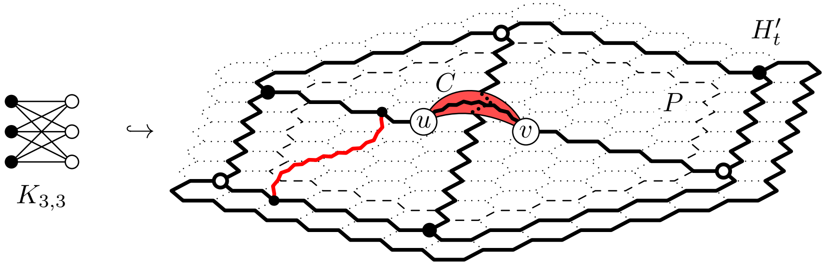

- Let $h(H_t)$ has no large component.
- There exists S such that $G \circ S$ is planar and $|S| \leq k$.
- A small component is bad if it contains an edge of S.
- A cell of $h(H_t)$ is called bad if
  - it has bad component attached to it, or
  - an edge of the cell is contracted by S.
- We have at most $7k$ bad cells, so for t large enough there exists $H_r \subset H_t$ such that $h(H_r)$ avoids all of them.
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- Let $h : H_r \hookrightarrow G$ be flat and S be inclusion-minimal.
- Consider embedding of $G' = G \circ S$.
- There is a ring R avoiding edges of S, separating I' and O' in G'.

Let I be de-contraction of I', so

$$I' = I \circ S.$$

Lemma

If S is inclusion-minimal, then

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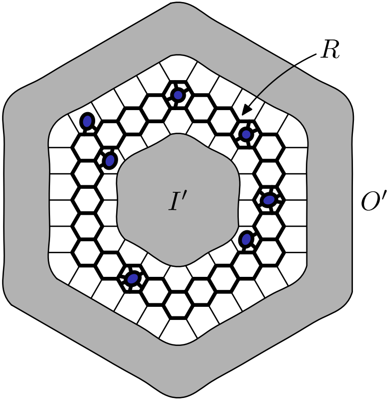

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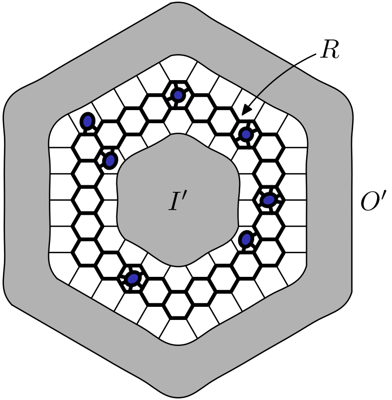

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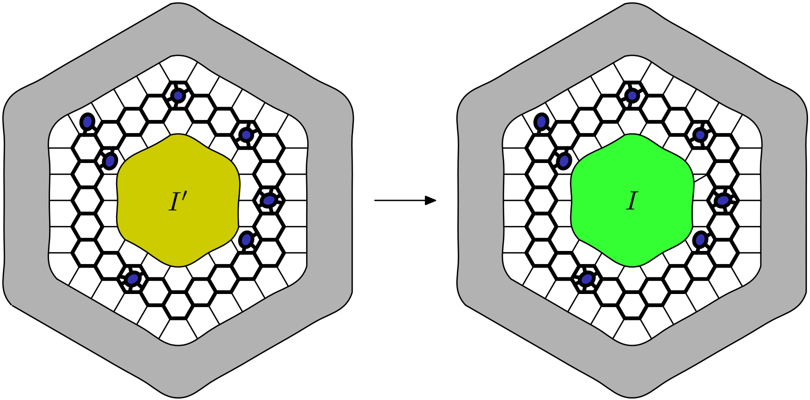

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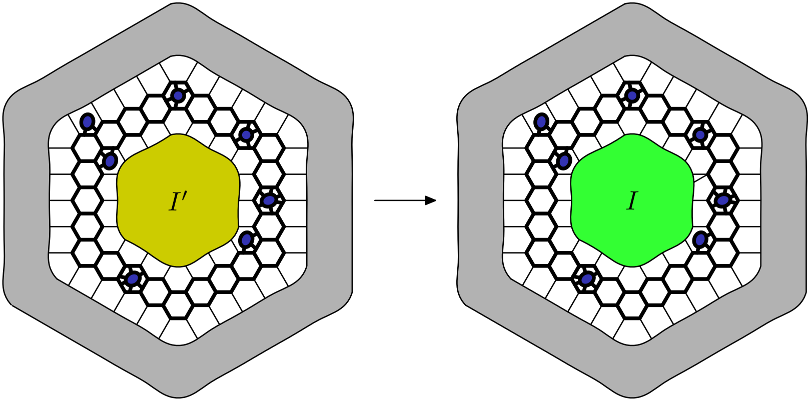

Phase I: Repeat until tree-width of G is large.

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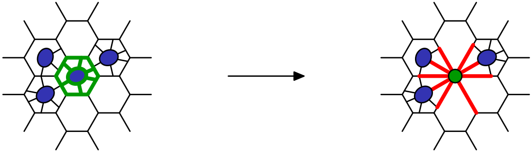

Phase II: Use Courcell's Theorem for small tree-width.

- The formula $\mu_H(S, G)$ is true if and only if H is a minor of $G \circ S$.
- The formula $\sigma(F, S)$ tests whether $F \cap S = \emptyset$.
- Then we just need to test whether φ holds for reduced graph G:

$$\varphi(S, G, F) := \sigma(F, S) \wedge P(S, G) \wedge \neg\mu_{K_5}(S, G) \wedge \neg\mu_{K_{3,3}}(S, G), .$$

- This can be done using Courcell's Theorem in time $\mathcal{O}(n \cdot g(k))$.

The total complexity is $\mathcal{O}(n^2 \cdot f(k))$.

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